

p -HAHN SEQUENCE SPACE

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ABSTRACT. The main purpose of the present paper is to introduce the space h_p and study of some properties of new sequence space. Also we compute their dual spaces and characterizations of some matrix transformations.

1. INTRODUCTION

By $\omega = \mathbb{C}^{\mathbb{N}}$, we denote the space of all real- or complex-valued sequences, where \mathbb{C} denotes the complex field and $\mathbb{N} = \{0, 1, 2, \dots\}$. Each linear subspace of ω is called a *sequence space*. For $x = (x_k) \in \omega$, we shall employ the sequence spaces $\ell_\infty = \{x : \sup_k |x_k| < \infty\}$, $c = \{x : \lim_k x_k \text{ exists}\}$, $c_0 = \{x : \lim_k x_k = 0\}$, $bs = \{x : \sup_n |\sum_{k=1}^n x_k| < \infty\}$, $cs = \{x : (\sum_{k=1}^n x_k) \in c\}$ and $\ell_p = \{x : \sum_k |x_k|^p < \infty, 1 \leq p < \infty\}$ which are Banach space with the following norms; $\|x\|_{\ell_\infty} = \sup_k |x_k|$, $\|x\|_{bs} = \|x\|_{cs} = \sup_n |\sum_{k=1}^n x_k|$ and $\|x\|_{\ell_p} = (\sum_k |x_k|^p)^{1/p}$ as usual, respectively. And also

$$bv^p = \left\{ x = (x_k) \in \omega : \sum_{k=1}^{\infty} |x_k - x_{k-1}|^p < \infty \right\},$$

$$\int \lambda = \{x = (x_k) \in \omega : (kx_k) \in \lambda\}.$$

A sequence, whose k -th term is x_k , is denoted by x or (x_k) . A *coordinate space* (or K -space) is a vector space of numerical sequences, where addition and scalar multiplication are defined pointwise. That is, a sequence space λ with a linear topology is called a K -space provided each of the maps $p_i : \lambda \rightarrow \mathbb{C}$ defined by $p_i(x) = x_i$ is continuous for all $i \in \mathbb{N}$. A BK -space is a K -space, which is also a Banach space with continuous coordinate functionals $f_k(x) = x_k$, ($k = 1, 2, \dots$). A K -space λ is called an FK -space provided λ is a complete linear metric space. An FK -space whose topology is normable is called a BK -space. If a normed sequence space λ contains a sequence (b_n) with the property that for every $x \in \lambda$ there is unique sequence of scalars (α_n) such that

$$\lim_{n \rightarrow \infty} \|x - (\alpha_0 b_0 + \alpha_1 b_1 + \dots + \alpha_n b_n)\| = 0$$

then (b_n) is called *Schauder basis* (or briefly basis) for λ . The series $\sum \alpha_k b_k$ which has the sum x is then called the expansion of x with respect to (b_n) , and written as $x = \sum \alpha_k b_k$. An FK -space λ is said to have AK property, if $\phi \subset \lambda$ and $\{e^k\}$ is a basis for λ , where e^k is a sequence whose only non-zero term is a 1 in k^{th} place for each $k \in \mathbb{N}$ and $\phi = \text{span}\{e^k\}$, the set of all finitely non-zero sequences. If ϕ is dense in λ , then λ is called an AD -space, thus AK implies AD .

Let λ and μ be two sequence spaces, and $A = (a_{nk})$ be an infinite matrix of complex numbers a_{nk} , where $k, n \in \mathbb{N}$. Then, we say that A defines a matrix mapping from λ into μ , and we denote it by writing $A : \lambda \rightarrow \mu$ if for every sequence $x = (x_k) \in \lambda$. The sequence $Ax = \{(Ax)_n\}$, the A -transform of x , is in μ ; where

$$(1.1) \quad (Ax)_n = \sum_k a_{nk} x_k \quad \text{for each } n \in \mathbb{N}.$$

For simplicity in notation, here and in what follows, the summation without limits runs from 0 to ∞ . By $(\lambda : \mu)$, we denote the class of all matrices A such that $A : \lambda \rightarrow \mu$. Thus, $A \in (\lambda : \mu)$ if and only if the series on the right side of (1.1) converges for each $n \in \mathbb{N}$ and each $x \in \lambda$ and we have $Ax = \{(Ax)_n\}_{n \in \mathbb{N}} \in \mu$ for all $x \in \lambda$. A sequence x is said to be A -summable to l if Ax converges to l which is called the A -limit of x .

The matrix domain λ_A of an infinite matrix A in a sequence space λ is defined by

$$(1.2) \quad \lambda_A = \{x = (x_k) \in \omega : Ax \in \lambda\}$$

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which is a sequence space (for several examples of matrix domains, see [2] p. 49-176). In [5], Başar and Altay have defined the sequence space bv_p which consists of all sequences such that Δ -transforms of them are in ℓ_p where Δ denotes the matrix $\Delta = (\delta_{nk})$

$$\Delta = \delta_{nk} = \begin{cases} (-1)^{n-k} & , \quad (n-1 \leq k \leq n) \\ 0 & , \quad (0 \leq k < n-1 \text{ or } k > n) \end{cases}$$

for all $k, n \in \mathbb{N}$. The space $[\ell(p)]_{A^u} = bv(u, p)$ has been studied by Başar et al. [3] where

$$A^u = a_{nk}^u = \begin{cases} (-1)^{n-k} u_k & , \quad (n-1 \leq k \leq n) \\ 0 & , \quad (0 \leq k < n-1 \text{ or } k > n) \end{cases}$$

for all $k, n \in \mathbb{N}$.

In the present paper, we introduce p -Hahn sequence space. We investigate its some properties and compute duals of this space and characterized some matrix transformations.

We assume throughout that $p^{-1} + q^{-1} = 1$ for $p, q \geq 1$. We denote the collection of all finite subsets of \mathbb{N} be \mathcal{F} .

2. NEW HAHN SEQUENCE SPACE

Hahn [7] introduced the BK -space h of all sequences $x = (x_k)$ such that

$$h = \left\{ x : \sum_{k=1}^{\infty} k|\Delta x_k| < \infty \text{ and } \lim_{k \rightarrow \infty} x_k = 0 \right\},$$

where $\Delta x_k = x_k - x_{k+1}$, for all $k \in \mathbb{N}$. The following norm

$$\|x\|_h = \sum_k k|\Delta x_k| + \sup_k |x_k|$$

was defined on the space h by Hahn [7] (and also [6]). Rao ([11], Proposition 2.1) defined a new norm on h as $\|x\| = \sum_k k|\Delta x_k|$. Goes and Goes [6] proved that the space h is a BK -space.

Hahn proved following properties of the space h :

- Lemma 2.1.** (i) h is a Banach space.
(ii) $h \subset \ell_1 \cap \int c_0$.
(iii) $h^\beta = \sigma_\infty$.

In [6], Goes and Goes studied functional analytic properties of the BK -space $bv_0 \cap d\ell_1$. Additionally, Goes and Goes considered the arithmetic means of sequences in bv_0 and $bv_0 \cap d\ell_1$, and used an important fact which the sequence of arithmetic means $(n^{-1} \sum_{k=1}^n x_k)$ of an $x \in bv_0$ is a quasiconvex null sequence. And also Goes and Goes proved that $h = \ell_1 \cap \int bv = \ell_1 \cap \int bv_0$.

Rao [11] studied some geometric properties of Hahn sequence space and gave the characterizations of some classes of matrix transformations.

Balasubramanian and Pandiarani[1] defined the new sequence space $h(F)$ called the Hahn sequence space of fuzzy numbers and proved that β - and γ -duals of $h(F)$ is the Cesàro space of the set of all fuzzy bounded sequences.

Kirişci [8] compiled to studies on Hahn sequence space and defined a new Hahn sequence space by Cesàro mean in [9].

Now, we introduce the sequence space h_p by

$$h_p = \left\{ x : \sum_{k=1}^{\infty} (k|\Delta x_k|)^p < \infty \text{ and } \lim_{k \rightarrow \infty} x_k = 0 \right\} \quad (1 < p < \infty)$$

where $\Delta x_k = (x_k - x_{k+1})$, $(k = 1, 2, \dots)$. If we take $p = 1$, $h_p = h$ which called Hahn sequence spaces.

Define the sequence $y = (y_k)$, which will be frequently used, by the M -transform of a sequence $x = (x_k)$, i.e.,

$$(2.1) \quad y_k = (Mx)_k = k(x_k - x_{k+1}).$$

where $M = (m_{nk})$ with

$$(2.2) \quad m_{nk} = \begin{cases} n & , & (n = k) \\ -n & , & (n + 1 = k) \\ 0 & , & other \end{cases}$$

for all $k, n \in \mathbb{N}$.

Theorem 2.2. $h_p = \ell_p \cap \int bv^p = \ell_p \cap \int bv_0^p$

Proof. We consider

$$k\Delta x_k \leq x_k + \Delta(kx_k).$$

Then, for $x \in \ell_p \cap \int bv^p$

$$\sum_{k=1}^n k|\Delta x_k| \leq \sum_{k=1}^n |x_k| + \sum_{k=1}^n |\Delta(kx_k)|$$

and from $|a + b|^p \leq 2^p(|a|^p + |b|^p)$, $(1 \leq p < \infty)$, we obtain

$$\sum_{k=1}^n k^p |\Delta x_k|^p \leq 2^p \left[\sum_{k=1}^n |x_k|^p + \sum_{k=1}^n |\Delta(kx_k)|^p \right].$$

For each positive integer r , we get

$$\sum_{k=1}^r k^p |\Delta x_k|^p \leq 2^p \left[\sum_{k=1}^r |x_k|^p + \sum_{k=1}^r |\Delta(kx_k)|^p \right].$$

and as $r \rightarrow \infty$

$$\sum_{k=1}^{\infty} k^p |\Delta x_k|^p \leq 2^p \left[\sum_{k=1}^{\infty} |x_k|^p + \sum_{k=1}^{\infty} |\Delta(kx_k)|^p \right].$$

and $\lim_{k \rightarrow \infty} x_k = 0$. Then $x \in h_p$ and

$$(2.3) \quad \ell_p \cap \int bv^p \subset h_p.$$

Let $x \in h_p$ and we consider

$$\sum_{k=1}^{\infty} |x_{k+1}|^p - \sum_{k=1}^{\infty} |\Delta(kx_k)|^p \leq \sum_{k=1}^{\infty} k^p |\Delta x_k|^p.$$

The series $\sum_{k=1}^{\infty} |x_{k+1}|^p$ is convergent from the definition of ℓ_p . Also $\sum_{k=1}^{\infty} |\Delta(kx_k)|^p < \infty$ and therefore $x \in \ell_p \cap \int bv^p$. Then

$$(2.4) \quad h_p \subset \ell_p \cap \int bv^p.$$

Form (2.3) and (2.4), we obtain $h_p = \ell_p \cap \int bv^p$. □

Theorem 2.3. *The sequence space h_p is a BK-space with AK.*

Proof. If x is any sequence, we write $\sigma_n(x) = M_n x$. Let $\varepsilon > 0$ and $x \in h_p$ be given. Then there exists N such that

$$(2.5) \quad |\sigma_n(x)| < \varepsilon/2$$

for all $n \geq N$. Now let $m \geq N$ be given. Then we have for all $n \geq m + 1$ by (2.5)

$$\left| \sigma_n(x - x^{[m]}) \right| \leq \left[\sum_{k=m+1}^{\infty} \left| k(\Delta x_k) \right|^p \right]^{1/p} \leq |\sigma_n(x)| + |\sigma_m(x)| < \varepsilon/2 + \varepsilon/2 = \varepsilon$$

whence $\|x - x^{[m]}\|_{h_p} \leq \varepsilon$ for all $m \geq N$. This shows $x = \lim_{m \rightarrow \infty} x^{[m]}$. □

Since h_p is an AK-space and every AK-space is AD, we can give the following corollary:

Corollary 2.4. *The sequence space h_p has AD.*

Theorem 2.5. Define a sequence $b^{(k)} = \{b_n^{(k)}\}_{n \in \mathbb{N}}$ of elements of the space h_p for every fixed $k \in \mathbb{N}$ by

$$b_n^{(k)} = \begin{cases} \frac{1}{k} & , \quad (n \leq k) \\ 0 & , \quad (n > k) \end{cases}$$

Then the sequence $\{b_n^{(k)}\}_{n \in \mathbb{N}}$ is a basis for the space h_p , and any $x \in h_p$ has a unique representation of the form

$$(2.6) \quad x = \sum_k \lambda_k b^{(k)}$$

where $\lambda_k = (Mx)_k$ for all $k \in \mathbb{N}$ and $1 \leq p < \infty$.

Proof. It is clear that $\{b^{(k)}\} \subset h_p$, since

$$(2.7) \quad Mb^{(k)} = e^k \in \ell_1, \quad (k = 0, 1, 2, \dots).$$

$1 \leq p < \infty$. Let $x \in h_p$ be given. For every non-negative integer m , we put

$$(2.8) \quad x^{[m]} = \sum_{k=0}^m \lambda_k b^{(k)}.$$

Then, we obtain by applying M to (2.8) with (2.7) that

$$Mx^{[m]} = \sum_{k=0}^m \lambda_k Mb^{(k)} = \sum_{k=0}^m (Mx)_k e^k$$

and

$$\left\{ M(x - x^{[m]}) \right\}_i = \begin{cases} 0 & , \quad (0 \leq i \leq m) \\ (Mx)_i & , \quad (i > m) \end{cases} ; \quad (i, m \in \mathbb{N}).$$

Given $\varepsilon > 0$, then there is an integer m_0 such that

$$\left[\sum_{i=m}^{\infty} |i \cdot (\Delta x)_i|^p \right]^{1/p} < \frac{\varepsilon}{2}$$

for all $m \geq m_0$. Hence,

$$\|x - x^{[m]}\|_{h_p} = \left[\sum_{i=m}^{\infty} |i \cdot (\Delta x)_i|^p \right]^{1/p} \leq \left[\sum_{i=m_0}^{\infty} |i \cdot (\Delta x)_i|^p \right]^{1/p} < \frac{\varepsilon}{2} < \varepsilon$$

for all $m \geq m_0$ which proves that $x \in h_p$ is represented as in (2.6).

To show the uniqueness of this representation, we assume that $x = \sum_k \mu_k b^{(k)}$. Now, we define the transformation T with the notation of (2.1), from h_p to ℓ_p by $x \mapsto y = Tx$. The linearity of T is clear. Since the linear transformation T is continuous we have at this stage that

$$(Mx)_n = \sum_k \mu_k \{Mb^{(k)}\}_n = \sum_k \mu_k e_n^k = \mu_n; \quad (n \in \mathbb{N})$$

which contradicts the fact that $(Mx)_n = \lambda_n$ for all $n \in \mathbb{N}$. Hence, the representation (2.6) of $x \in h_p$ is unique. \square

Theorem 2.6. Except the case $p = 2$, the space h_p is not an inner product space, therefore not a Hilbert space for $1 < p < \infty$.

Proof. For $p = 2$, we will show that the space h_2 is a Hilbert space. Since the space h_p is a BK -space from Theorem 2.3, the space h_2 is a BK -space, for $p = 2$. Also its norm can be obtained from an inner product, i.e., $\|x\|_{h_2} = \langle k\Delta x, k\Delta x \rangle^{1/2}$ holds. Then the space h_2 is a Hilbert space.

Now consider the sequences $e_1 = (1, 0, 0, 0, \dots)$ and $e_2 = (0, 1, 0, 0, \dots)$. Then we see that $\|e_1 + e_2\|_{h_p}^2 + \|e_1 - e_2\|_{h_p}^2 \neq 2 \cdot (\|e_1\|_{h_p}^2 + \|e_2\|_{h_p}^2)$, i.e., the norm of the space h_p does not satisfy the parallelogram equality, which means that the norm cannot be obtained from inner product. Hence, the space h_p with $p \neq 2$ is a Banach space that is not a Hilbert space. \square

Now, we give some inclusion relations concerning with the space h_p .

Theorem 2.7. Neither of the spaces h_p and ℓ_∞ includes the other one, where $1 < p < \infty$.

Proof. Now we choose the sequences $a = (a_k)$ and $b = (b_k)$ such that $a = (a_k) = \{(-1)^k\}$ and $b = (b_k) = \sum_{i=1}^k 1/(i+1)$. The sequence $a = (a_k)$ is in $\ell_\infty \setminus h_p$ and the sequence $b = (b_k)$ is in $h_p \setminus \ell_\infty$. So, the sequences h_p and ℓ_∞ does not include each other. \square

Theorem 2.8. *If $1 \leq p < r$, then $h_p \subset h_r$.*

Proof. This can be obtained by analogy with the proof of Theorem 2.6 in [5]. So, we omit the details. \square

3. DUALS OF NEW HAHN SEQUENCE SPACE

In this section, we state and prove the theorems determining the α -, β - and γ -duals of the sequence space h_p .

Let x and y be sequences, X and Y be subsets of ω and $A = (a_{nk})_{n,k=0}^\infty$ be an infinite matrix of complex numbers. We write $xy = (x_k y_k)_{k=0}^\infty$, $x^{-1} * Y = \{a \in \omega : ax \in Y\}$ and $M(X, Y) = \bigcap_{x \in X} x^{-1} * Y = \{a \in \omega : ax \in Y \text{ for all } x \in X\}$ for the *multiplier space* of X and Y . In the special cases of $Y = \{\ell_1, cs, bs\}$, we write $x^\alpha = x^{-1} * \ell_1$, $x^\beta = x^{-1} * cs$, $x^\gamma = x^{-1} * bs$ and $X^\alpha = M(X, \ell_1)$, $X^\beta = M(X, cs)$, $X^\gamma = M(X, bs)$ for the α -dual, β -dual, γ -dual of X . By $A_n = (a_{nk})_{k=0}^\infty$ we denote the sequence in the n -th row of A , and we write $A_n(x) = \sum_{k=0}^\infty a_{nk} x_k$ $n = (0, 1, \dots)$ and $A(x) = (A_n(x))_{n=0}^\infty$, provided $A_n \in x^\beta$ for all n .

Given an FK -space X containing ϕ , its conjugate is denoted by X' and its f -dual or sequential dual is denoted by X^f and is given by $X^f = \{ \text{all sequences } (f(e^k)) : f \in X' \}$.

Let λ be a sequence space. Then λ is called *perfect* if $\lambda = \lambda^{\alpha\alpha}$; *normal* if $y \in \lambda$ whenever $|y_k| \leq |x_k|$, $k \geq 1$ for some $x \in \lambda$; *monotone* if λ contains the canonical preimages of all its stepspace.

Lemma 3.1. (i). $A \in (h : \ell_1)$ if and only if

$$(3.1) \quad \sum_{n=1}^{\infty} |a_{nk}| \text{ converges, } (k = 1, 2, \dots)$$

$$(3.2) \quad \sup_k \frac{1}{k} \sum_{n=1}^{\infty} \left| \sum_{v=1}^k a_{nv} \right| < \infty.$$

(ii). $A \in (\ell_p : \ell_1)$ if and only if

$$\sup_{K \in \mathcal{F}} \sum_k \left| \sum_{n \in K} a_{nk} \right|^q < \infty$$

Lemma 3.2. (i). $A \in (h : c)$ if and only if

$$(3.3) \quad \sup_{n,k} \frac{1}{k} \left| \sum_{v=1}^k a_{nv} \right| < \infty$$

$$(3.4) \quad \lim_{n \rightarrow \infty} a_{nk} \text{ exists, } (k = 1, 2, \dots)$$

(ii). $A \in (\ell_p : c)$ if and only if (3.4) holds and

$$(3.5) \quad \sup_n \sum_k |a_{nk}|^q < \infty, \quad 1 < p < \infty$$

Lemma 3.3. (i). $A \in (h : \ell_\infty)$ if and only if (3.3) holds.

(ii). $A \in (\ell_p : \ell_\infty)$ if and only if (3.5) holds with $1 < p \leq \infty$.

Lemma 3.4. $A \in (h : c_0)$ if and only if (3.3) holds and

$$(3.6) \quad \lim_{n \rightarrow \infty} a_{nk} = 0$$

Lemma 3.5. $A \in (h : h)$ if and only if (3.6) holds and

$$(3.7) \quad \sum_{n=1}^{\infty} n |a_{nk} - a_{n+1,k}| \text{ converges, } (k = 1, 2, \dots)$$

$$(3.8) \quad \sup_k \frac{1}{k} \sum_{n=1}^{\infty} n \left| \sum_{v=1}^k (a_{nv} - a_{n+1,v}) \right| < \infty.$$

Theorem 3.6. We define the sets d_1 and d_2 as follows:

$$\begin{aligned} d_1 &= \{a = (a_k) \in \omega : \sup_{K \in \mathcal{F}} \sum_k \left| \sum_{n \in K} \frac{1}{k} a_n \right|^q < \infty\} \quad 1 < p < \infty \\ d_2 &= \{a = (a_k) \in \omega : \sup_{K \in \mathcal{F}} \sum_k \left| \sum_{n \in K} \frac{1}{k} a_n \right| < \infty\}. \end{aligned}$$

Then $[h_p]^\alpha = d_1$ and $[h]^\alpha = d_2$.

Proof. We give the proof only for the case $[h_p]^\alpha = d_1$. Let us take any $a = (a_k) \in \omega$ and consider the equation

$$(3.9) \quad a_n x_n = \sum_{j=n}^{\infty} \frac{a_n}{j} y_j = (Dy)_n \quad (n \in \mathbb{N})$$

where $D = (d_{nk})$ is defined by

$$d_{nk} = \begin{cases} \frac{a_n}{k} & , \quad k \geq n \\ 0 & , \quad k < n \end{cases}$$

for all $k, n \in \mathbb{N}$. It follows from (3.9) with Lemma 3.1(ii) that $ax = (a_n x_n) \in \ell_1$ whenever $x = (x_k) \in h_p$ if and only if $Dy \in \ell_1$ whenever $y = (y_k) \in \ell_p$. This means that $a = (a_n) \in [h_p]^\alpha$ whenever $x = (x_n) \in h_p$ if and only if $D \in (h_p : \ell_1)$. This gives the result that $[h_p]^\alpha = d_1$. \square

Hahn[7] proved that $[h]^\beta = \sigma_\infty$ where $\sigma_\infty = \{a = (a_k) \in \omega : \sup_n \frac{1}{n} |\sum_{k=1}^n a_k| < \infty\}$. We can give β -dual of h_p .

Theorem 3.7. Let $1 < p < \infty$. Then, $[h_p]^\beta = d_3$ where

$$d_3 = \left\{ a = (a_k) \in \omega : \sup_{n \in \mathbb{N}} (n^{-1})^q \sum_k \left| \sum_{j=k}^n a_j \right|^q < \infty \right\}$$

Proof. Consider the equation

$$(3.10) \quad \sum_{k=1}^n a_k x_k = \sum_{k=1}^n a_k \left(\sum_{j=k}^n \frac{y_j}{j} \right) = \sum_{k=1}^n \left(\sum_{j=1}^k \frac{a_j}{k} \right) y_k = (By)_n \quad (n \in \mathbb{N});$$

where $B = (b_{nk})$ are defined by

$$b_{nk} = \begin{cases} \sum_{j=1}^k \frac{a_j}{k} & , \quad (n \leq k) \\ 0 & , \quad (n > k) \end{cases}$$

for all $k, n \in \mathbb{N}$. Thus we deduce from Lemma 3.2 (ii) with (3.10) that $ax = (a_k x_k) \in cs$ whenever $x = (x_k) \in h_p$ if and only if $By \in c$ whenever $y = (y_k) \in \ell_p$. Thus, $(a_k) \in cs$ and $(a_k) \in d_3$ by (3.4) and (3.5), respectively. Nevertheless, the inclusion $d_3 \subset cs$ holds and, thus, we have $(a_k) \in d_3$ whence $[h_p]^\beta = d_3$. \square

Lemma 3.8. ([12], Theorem 7.2.7) Let X be an FK -space with $X \supset \phi$. Then,

- (i) $X^\beta \subset X^\gamma \subset X^f$;
- (ii) If X has AK , $X^\beta = X^f$;
- (iii) If X has AD , $X^\beta = X^\gamma$.

From Theorem 2.3, Corollary 2.4 and Lemma 3.8, we can write the following corollary:

Corollary 3.9. (i) $[h_p]^\beta = [h_p]^f$
(ii) $[h_p]^\beta = [h_p]^\gamma$.

Lemma 3.10. Let λ be a sequence space. Then the following assertions are true:

- (i) λ is perfect $\Rightarrow \lambda$ is normal $\Rightarrow \lambda$ is monotone;
- (ii) λ is normal $\Rightarrow \lambda^\alpha = \lambda^\gamma$;
- (iii) λ is monotone $\Rightarrow \lambda^\alpha = \lambda^\beta$.

Combining Theorem 3.6, Theorem 3.7 and Lemma 3.10, we can give the following corollary:

Corollary 3.11. The space h_p is not monotone and so it is neither normal nor perfect.

4. MATRIX TRANSFORMATIONS

In this section, we characterize some matrix transformations on the space h_p .

Lemma 4.1. [5] *Let λ, μ be any two sequence spaces, A be an infinite matrix and U a triangle matrix matrix. Then, $A \in (\lambda : \mu_U)$ if and only if $UA \in (\lambda : \mu)$.*

If we define $\tilde{a}_{nk} = n(a_{nk} - a_{n+1,k})$, then we can give following corollary from Lemma 4.1 with $U = M$ defined by (2.2):

Corollary 4.2. (i) $A \in (\ell_1 : h)$ if and only if

$$\sup_k \sum_n |\tilde{a}_{nk}| < \infty$$

(ii) $A \in (c : h) = (c_0 : h) = (\ell_\infty : h)$ if and only if

$$\sup_{K \in \mathcal{F}} \sum_n \left| \sum_{k \in K} \tilde{a}_{nk} \right| < \infty$$

Theorem 4.3. *Suppose that the entries of the infinite matrices $A = (a_{nk})$ and $E = (e_{nk})$ are connected with the relation*

$$(4.1) \quad e_{nk} = \bar{a}_{nk}$$

for all $k, n \in \mathbb{N}$, where $\bar{a}_{nk} = \sum_{j=k}^{\infty} \frac{a_{nj}}{j}$ and μ be any sequence space. Then $A \in (h_p : \mu)$ if and only if $\{a_{nk}\}_{k \in \mathbb{N}} \in [h_p]^\beta$ for all $n \in \mathbb{N}$ and $E \in (h : \mu)$.

Proof. Let μ be any given sequence spaces. Suppose that (4.1) holds between $A = (a_{nk})$ and $E = (e_{nk})$, and take into account that the spaces h_p and h are norm isomorphic.

Let $A \in (h_p : \mu)$ and take any $y = y_k \in h$. Then EM exists and $\{a_{nk}\}_{k \in \mathbb{N}} \in [h_p]^\beta$ which yields that $\{e_{nk}\}_{k \in \mathbb{N}} \in \ell_1$ for each $n \in \mathbb{N}$. Hence, Ey exists and thus

$$\sum_k e_{nk} y_k = \sum_k a_{nk} x_k$$

for all $n \in \mathbb{N}$. We have that $Ey = Ax$ which leads us to the consequence $E \in (h : \mu)$.

Conversely, let $\{a_{nk}\}_{k \in \mathbb{N}} \in d_1$ for all $n \in \mathbb{N}$ and $E \in (h : \mu)$ hold, and take any $x = x_k \in h_p$. Then, Ax exists. Therefore, we obtain from the equality

$$\sum_k a_{nk} x_k = \sum_k \left[\sum_{j=k}^{\infty} \frac{a_{nj}}{j} \right] y_k$$

for all $n \in \mathbb{N}$. Thus $Ax = Ey$ and this shows that $A \in (h_p : \mu)$. \square

If we use the Corollary 4.2 and change the roles of the spaces h_p with μ in Theorem ref4thm1, we can give following theorem:

Theorem 4.4. *Suppose that the entries of the infinite matrices $A = (a_{nk})$ and $\tilde{A} = (\tilde{a}_{nk})$ are connected with the relation $\tilde{a}_{nk} = n(a_{nk} - a_{n+1,k})$ for all $k, n \in \mathbb{N}$ and μ be any sequence space. Then $A \in (\mu : h_p)$ if and only if and $\tilde{A} \in (\mu : h)$.*

Proof. Let $z = (z_k) \in \mu$ and consider the following equality

$$\sum_{k=0}^m \tilde{a}_{nk} z_k = \sum_{k=0}^m n(a_{nk} - a_{n+1,k}) z_k \quad \text{for all, } m, n \in \mathbb{N}$$

which yields that as $m \rightarrow \infty$ that $(\tilde{A}z)_n = \{M(Az)\}_n$ for all $n \in \mathbb{N}$. Therefore, one can observe from here that $Az \in h_p$ whenever $z \in \mu$ if and only if $\tilde{A}z \in h$ whenever $z \in \mu$. \square

We can give following corollaries from Lemma 3.1-3.5, Corollary 4.2, Theorem 4.3 and Theorem 4.4:

Corollary 4.5. (i) $A \in (h_p : \ell_\infty)$ if and only if $\{a_{nk}\}_{k \in \mathbb{N}} \in [h_p]^\beta$ for all $n \in \mathbb{N}$ and

$$(4.2) \quad \sup_k \left(\frac{1}{k} \left| \sum_{v=1}^k \bar{a}_{nv} \right| \right)^q < \infty$$

- (ii) $A \in (h_p : c)$ if and only if $\{a_{nk}\}_{k \in \mathbb{N}} \in [h_p]^\beta$, (4.2) holds and
- (4.3) $\lim_{n \rightarrow \infty} \bar{a}_{nk} = \alpha_k \quad (k \in \mathbb{N})$
- (iii) $A \in (h_p : c_0)$ if and only if $\{a_{nk}\}_{k \in \mathbb{N}} \in [h_p]^\beta$, (4.2) holds and (4.3) holds with $\alpha_k = 0$.
- (iv) $A \in (h_p : \ell_1)$ if and only if $\{a_{nk}\}_{k \in \mathbb{N}} \in [h_p]^\beta$ and

$$\sum_{n=1}^{\infty} |\bar{a}_{nk}|^q \text{ converges, } (k = 1, 2, \dots)$$

$$\sup_k \frac{1}{k^q} \sum_{n=1}^{\infty} \left| \sum_{v=1}^k \bar{a}_{nv} \right|^q < \infty.$$

Corollary 4.6. (i) $A \in (\ell : h_p)$ if and only if

$$\sup_{K \in \mathcal{F}} \sum_k \left| \sum_{n \in K} \tilde{a}_{nk} \right| < \infty$$

- (ii) $A \in (c : h_p) = (c_0 : h_p) = (\ell_\infty : h_p)$ if and only if

$$\sup_{K \in \mathcal{F}} \sum_n \left| \sum_{k \in K} \tilde{a}_{nk} \right| < \infty$$

5. CONCLUSION

Hahn [7] defined the space h and gave some properties. Goes and Goes [6] studied its different properties. Rao [11] introduced the Hahn sequence space and investigated some properties in Banach space theory. Kirişci [8] compiled to studies of Hahn sequence space and defined a new Hahn sequence space by Cesàro mean in [9].

In this paper, we defined the space p -Hahn sequence spaces and gave some properties. In section 3, we compute the duals of the space h_p and characterize some matrix transformations related to this space, in section 4.

Finally, we should note that, as a natural continuation of the present paper, one can study the paranormed Hahn sequence space. Also it can be obtained the new Hahn sequence space by using Euler mean, Riesz mean, generalized weighted mean etc.

REFERENCES

- [1] T. Balasubramanian, A. Pandiarani, *The Hahn sequence spaces of fuzzy numbers*, Tamsui Oxf. J. Inf. Math. Sci. **27**(2), (2011), 213–224.
- [2] F. Başar, *Summability Theory and its Applications*, Bentham Science Publishers, e-books, Monographs, (2011).
- [3] F. Başar, B. Altay and M. Mursaleen, *Some generalizations of the space bv_p of p -bounded variation sequences*, Nonlinear Analysis, **68** (2008), 273–287.
- [4] B. Altay and F. Başar, *Certain topological properties and duals of the domain of a triangle matrix in a sequence spaces*, J. Math. Analysis and Appl., **336** (2007), 632–645.
- [5] F. Başar and B. Altay *On the space of sequences of p -bounded variation and related matrix mappings*, Ukrainian Math. J., **55**(1) (2003), 136–147.
- [6] G. Goes and S., Goes, *Sequences of bounded variation and sequences of Fourier coefficients I*, Math. Z., **118**(1970), 93–102.
- [7] H. Hahn, *Über Folgen linearer Operationen*, Monatsh. Math., **32**(1922), 3–88.
- [8] M. Kirişci *A survey on the Hahn sequence spaces*, Gen. Math. Notes, 19(2), 2013.
- [9] M. Kirişci *The Hahn sequence spaces sefined by Cesàro Mean*, Abstract and Applied Analysis, vol. 2013, Article ID 817659, 6 pages, 2013. doi:10.1155/2013/817659
- [10] S. A. Rakov, *Banach-Saks property of a Banach space*, Mat. Zametki, **26**(6)(1979), 823–834.
- [11] W. Chandrasekhara Rao, *The Hahn sequence spaces I*, Bull. Calcutta Math. Soc. **82**(1990), 72–78.
- [12] A. Wilansky, *Summability through Functinal Analysis*, North Holland, New York, (1984).

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